Connected spaces in which all connected sets containing some fixed point are closed.

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Notations.

Let X be a connected T_1-space and x_0 some fixed point of X such that any connected set containing x_0 is closed.
The characters z, y, z, u, v, ... denote points of X.
For a topological space Y, we write Y = A + B if Y is the topological sum of its subspaces A and B.
We will frequently apply the following two wellknown lemma's, most often with Y = {z} for some $z \in Z$:

Lemma 1. If Z and $Y \subset Z$ are connected and $Z \setminus Y = A + B$, then $Y \cup A$ (and $Y \cup B$) is connected.

Lemma 2. If Z and $Y \subset Z$ are connected and C is a component of $Z \setminus Y$ then $Z \setminus C$ is connected.

Let < be the relation (partial order) defined on X by:

$$x_0 < y \quad \text{for all } y \in X \setminus \{x_0\}$$
$$x < y \quad \text{if } x \text{ separates } x_0 \text{ and } y$$

Then X and < have the following properties:

Proposition 1. The relation < is antisymmetric and transitive; i.e. is a partial order.

Proof. If x < y and y < x then there exist A, B, C, $D \subset X$ such that $X \setminus \{x\} = A + B$, $X \setminus \{y\} = C + D$, $x_0 \in A \cap B$, $y \in B$ and $x \in D$. Now $A \cup \{x\}$ is connected (lemma 1), contained in C + D, but meeting both C(in x_0) and D(in x). Contradiction.

If $x < y$ and $y < z$ then there exist A, B, X, $D \subset X$ such that $X \setminus \{x\} = A + B$, $X \setminus \{y\} = C + D$, $x_0 \in A \cap C$, $y \in B$ and $z \in D$. Since $D \cup \{y\}$ is connected (lemma 1) and intersects B (in y) $D \cup \{y\} \subset B$. Hence $x < z$. ∎

Proposition 2. <u>For each $x \in X$ $\{y \mid y < x\}$ is linearly ordered</u> (and wellordered by >, see 6)

Proof. Let $y < x$, $z < x$ but $y \not\leq z$ and $z \not\leq y$. Then there exist A, B, C, $D \subset X$ such that $X \setminus \{y\} = A + B$, $X \setminus \{z\} = C + D$, $x_0 \in A \cap C$ and $x \in B \cap D$, but $z \notin B$, $y \notin D$ and so $z \in A$ and $y \in C$. Since $D \cup \{z\}$ is connected (lemma 1), and intersects A (in z), but does not contain y, $D \cup \{z\} \subset A$. This is contradictory to $x \in D \setminus A$. ∎

Proposition 3. <u>If $x \in X$ and C is a component of $X \setminus \{x\}$ which does not contain x_0, then C is open in X and $C^- = C \cup \{x\}$.</u> (If $x_0 \in C$, then C is closed in X).

Proof. $X \setminus C$ is connected (lemma 2), contains x_0 and is hence closed. Now C cannot be closed in X, because X is connected. As C is closed in $X \setminus \{x\}$, only x can be another limitpoint of C. ∎

Proposition 4. <u>For any x $\{y \mid x \not\leq y\}$ is the component of $X \setminus \{x\}$ which contains x_0, and hence this set is closed. So its complement $\{y \mid x \leq y\}$ is connected and open.</u>(

Proof. By definition of < $\{y \mid x \not\leq y\}$ is the quasicomponent of x_0 in $X \setminus \{x\}$. If this set was not connected, then it would contain a component C of $X \setminus \{x\}$ which does not contain x_0. But this C is open in X (by 3) and closed in $X \setminus \{x\}$. Thus $\{y \mid x \not\leq y\}$ is not a quasicomponent.

The connectedness of $\{y \mid x \leq y\}$ follows from lemma 2.∎

Proposition 5a. <u>For each non empty linearly ordered $A \subset X$ there is a (unique) $x \in X$ such that $x = \inf A$.</u>

5b. <u>Each $y \in X \setminus \{x_0\}$ has an immediate predecessor, which will be denoted by y'. For this point y'</u>:

$$\{z \mid y \leq z\}^- = \{z \mid y \leq z\} \cup \{y'\}$$

Proof. (a) Let $A^* = \{z \mid \exists a \in A \quad a < z\}$. By 4 this set is open and hence not closed, as X is connected. Let x be a boundary point of A^*. At first we will show that $x < a$ for all $a \in A$ (or $a \in A^*$). Since $x \notin A^*$ we have $a \nless x$ for all $a \in A$. Suppose x and some $a \in A$ are not comparible. Then, by 2, x cannot be compared with any $a \in A$. But then again by 2, $\{y \mid x \leq y\} \cap A^* = \emptyset$. Since $\{y \mid x \leq y\}$ is open (see 4), $x \notin A^-$.
Contradiction.
Secondly assume that for some y $\quad x < y$ and $y < a$ for all $a \in A$. For $a \in A$ let C_a be the component of a in $X \setminus \{y\}$. Since, by 4, $\{z \mid a \leq z\} \subset C_a$, the family $\{C_a \mid a \in A\}$ has no disjoint members. Hence it has a connected union. This means that for some component C of $X \setminus \{y\}$ $\quad A \subset C$. By 3 $C^- = C \cup \{y\}$, but C does not contain x since $x < y$. This contradicts $x \in A^- \subset C^-$.
So if A has no smallest element then $x = \inf A$.
(b) Let $A = \{y\}$, $y' = x =$ the boundary point of $\{z \mid x \leq z\}$.∎

For each ordertype α, ordered by $<$, let α^* denote the ordertype of α, ordered by $>$. It follows immediately from 5a and 5b that for each $x \in X$ the set $\{y \mid y \leq x\}$ has ordertype α^* for some <u>ordinal</u> α. If A is a linearly ordered subset of X, and B is an infinite strictly increasing sequence, then by consequence B is cofinal with A. It follows from 4 and 5b that X cannot have maximal members.
Thus we proved:

Proposition 6. <u>Let A be a linearly ordered subset of X, with ordertype α. If A is bounded in X then $\alpha = \beta^*$ for some ordinal β. If A is not bounded in X then $\alpha = \sum_{n \in N} \beta_n^*$, for some suitable countable set of ordinals $\{\beta_1, \beta_2, \ldots\}$.</u>

We feel that the following facts deserve special attention

7. <u>Any point of $X \setminus \{x_0\}$ separates X in infinitely many components (as follows from 4).</u>

8. <u>Any connected space has a non-closed connected</u> (proper) <u>subset</u> (else it were a space like X, but $X \setminus \{x_0\}$ is non-closed and connected).

9. <u>ZARANKIEWICZ [2].</u> <u>If M is a connected separable metric space and D is the set of points $x \in M$ for which $M \setminus \{x\}$ has at least 3 components, then D is countable.</u> On the other hand M has continuously many points.

   Corollary. <u>X is not separable metric.</u>

<u>Example of a Hausdorffspace X.</u>

Let $\mathbb{N}$ be the set of natural numbers, and $P \subset \mathbb{N}$ the set of primenumbers.

Put $X = \bigcup \{\mathbb{N}^n \mid n \in \mathbb{N}\} \cup \{0\}$.

For $x \in X$ we define length $x = \begin{cases} 2 & \text{if } x = 0 \\ n+2 & \text{if } x \in \mathbb{N}^n. \end{cases}$

We define a partial order on X by taking $0 \leq x$ for all $x \in X$ and $x \leq y$ if x is an initial sequent of y, i.e. if $x \in \mathbb{N}^n$, $y \in \mathbb{N}^m$, $n \leq m$, and there exist $a_1, \ldots a_m \in \mathbb{N}$ such that $x = (a_1, \ldots a_n)$, $y = (a_1, \ldots a_m)$.
If $x = (a_1, \ldots a_n)$ then let $x' = (a_1, \ldots a_{n-1})$.
As a subbase for the open sets we take all sets

(i) $\{z \mid x \leq z\}$ for each $x \in X$

(ii) $\{z \mid x \nleq z \wedge z \neq x'\}$ for each $x \in X$

(iii) $\{z \mid$ the only primes deviding length y are $p_1, \ldots p_n\}$ for each finite set of primes, $p_1, \ldots p_n$.

10. <u>X is a Hausdorffspace</u>

    Let $u, v \in X$. We distinguish between

    (a) $u < v$ and even $u < v'$

    (b) neither $u < v$ nor $v < u$

    (c) $u = v'$.

(a) In this case $\{y \mid v \nleq y \wedge y \neq v'\}$ and $\{z \mid v \leq z\}$ are disjoint neighbourhoods of u and v.

(b) Now $\{z \mid u \leq z$ and $\{z \mid v \leq z\}$ are disjoint neighbourhoods of u and v, since $u < z$ and $v < z$ (for some $z \in X$) would imply that u and v are comparable (by definition of X).

(c) Let $p_1, \ldots p_n$ be the set of prime numbers which devide length u, and $q_1, \ldots q_m$ idem for length v. Then $\{p_1, \ldots p_n\} \cap \cap \{q_1, \ldots q_m\} = \emptyset$ and so
$\{z \mid \forall p \in P \quad p|\text{length } z \Rightarrow p \in \{p_1, \ldots p_n\}\}$ and
$\{z \mid \forall p \in P \quad p|\text{length } z \Rightarrow p \in \{q_1, \ldots q_m\}\}$
are disjoint neighbourhoods of u and v.∎

11. Any connected set $C \subset X$ containing 0 is closed.

If $u \in X \setminus C$ then we will show that C is disjoint from $\{y \mid u \leq y\}$; hence C is closed. Suppose $u \leq y$ for some $y \in U$, and $u \in \mathbb{N}^n$, $y = (a_1, \ldots a_m) \in \mathbb{N}^m$. Now

$C = (C \cap \{z \mid (a, \ldots a_{n+1}) \leq z\}) + (C \cap \{z \mid (a_1, \ldots a_{n+1}) \nleq z \wedge z \neq u\}$,

contradictory to the connectedness of C.∎

12. X is connected.

Lemma. For each $u \in X \setminus \{0\}$ the points u and u' have no disjoint closed neighbourhoods.

Proof of the connectedness of X.
Suppose $X = A + B$, $0 \in A$, $y \in B$ is such that length y is minimal. Then $y' \in A$, and A and B are disjoint closed neighbourhoods of y' and y. Contradiction.∎

Proof of the lemma. Let $u = (a_1, \ldots a_l)$.
For each point $x \in X$ and each finite family $\{x_1, \ldots x_n\}$ such that $x_i \nleq x$ and $x' \neq x_i'$ we define the following neighbourhood of x:

$$U(x,\{x_1, \ldots x_n\}) = \{z \mid x \leq z\} \cap \cap \{z \mid \forall p \in P\ (p|\text{length } z) \Rightarrow (p|\text{length } x)\} \cap \bigcap_{i=1}^{n} \{z \mid x_i \nleq z \wedge z \neq x_i'\}.$$

It should be clear that if the $x_1, \ldots x_n$ vary we obtain a neighbourhoodbase of x. (We may also vary only over those x_i for which $x < x_i'$).

For $x = (a_1, \ldots a_n)$ we let $\max x = \max\{a_1, \ldots a_n\}$.

Now let $U(u',\{x_1, \ldots x_n\})$ and $U(u,\{x_{n+1}, \ldots x_m\})$ be two arbitrary basic neighbourhood of u' and u.

Put

$$N = \max\{\max x_i \mid i=1, \ldots k\} + 1$$
$$L = (\text{length } x)(\text{length } x') - 2$$
$$v = (a_1, \ldots a_l, N, N, \ldots N) \in N^L.$$

We will show that $v \in U(u',\{x_1, \ldots x_n\})^- \cap U(u,\{x_{n+1}, \ldots x_m\})^-$.

Let $U(v,\{x_{m+1},\ldots\})$ be an arbitrary neighbourhood of v. Put

$$N' = \max\{\max x_i \mid i=1, \ldots k, \ldots m, m+1, \ldots\} + 1.$$

Let $p, q \in P$ be such that $p \mid \text{length } u'$, $q \mid \text{length } u$, and choose $r \in \mathbb{N}$ such that $p^r > L$ and $q^r > L$. Then

$$\underbrace{(\underbrace{a_1, \ldots a_l, N, \ldots N}_{L \text{ numbers}}, N', \ldots N')}_{p^r \text{ numbers}} \in U(u',\{x_1, \ldots x_n\}) \cap U(v,\{x_{m+1},\ldots\})$$

and

$$\underbrace{(\underbrace{a_1, \ldots a_l, N, \ldots N}_{L \text{ numbers}}, N', \ldots N')}_{q^r \text{ numbers}} \in U(u,\{x_{n+1} \ldots x_m\}) \cap U(v,\{x_{m+1},\ldots\})$$

It is easily seen that if $C \subset X$ is connected, then each $x \in C$ disconnects C, except maybe inf C (cf 11 and 4 and 5). In the terminology of [1]: each connected subset of X has at most one endpoint.

The points (1), (2), (3) are such that none of them separates the other 2. Zo this settles the problem mentioned in [1] p 24 remark 3.

REFERENCES

[1] H. Kok — On conditions equivalent to the orderability of a connected space.
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